Rogue Waves in the Ablowitz-Ladik Equations

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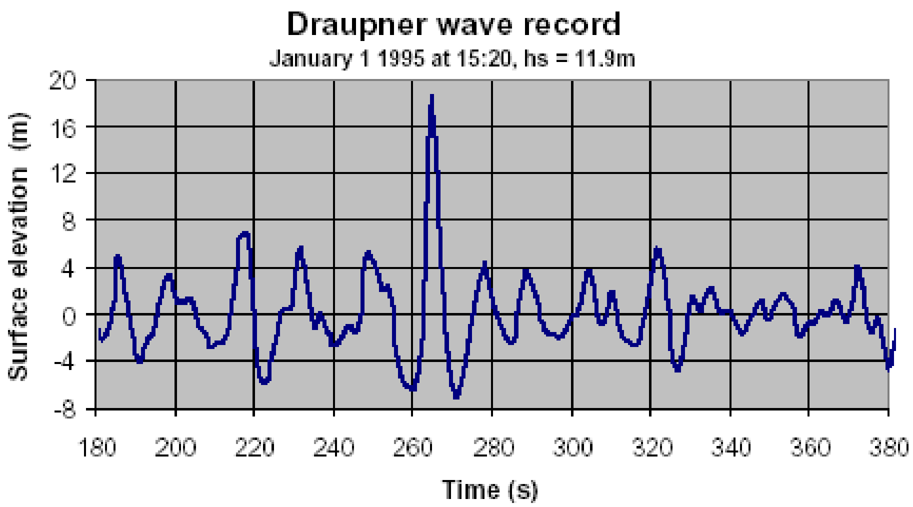
Talk Outline:

- Introduction
- Modulation Instability in the Ablowitz-Ladik Equations
- Formulae for General Rogue Waves in AL Equations
- Dynamics of General Rogue Waves in AL Equations
- Derivation of General Rogue Waves in AL Equations
- Comparison with Rogue Waves in Davey-Stewartson Equations
- Summary

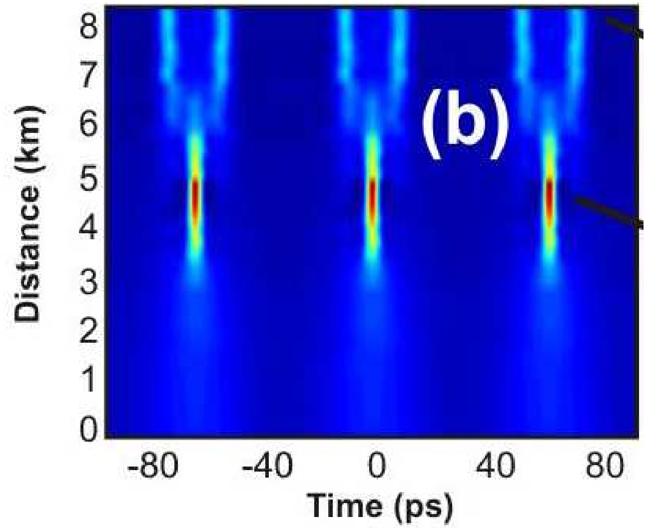
1. Introduction

Rogue waves are large and spontaneous waves in nonlinear systems.

Rogue waves in ocean



Optical rogue waves



Kibler et al. 2010

Rogue waves can be damaging.

Thus understanding of rogue waves and conditions for their appearance is necessary.

1. Introduction

A strong point of view: rogue waves are due to modulation instability.

Mathematically the simplest model for describing modulation instability in optics and water waves is the NLS equation:

$$iu_t = u_{xx} + 2|u|^2 u.$$

The simplest rogue wave in this equation was given by Peregrine (1983).

Higher-order rogue waves in this NLS equation were derived by

- **Akhmediev *et al.* (2009 --)**
- **Matveev et al (2010 --)**
- **Guo, Ling and Liu (2012)**
- **Ohta & Yang (2012)**
- **He et al. (2013)**
- **........**

1. Introduction

Notes:

- **The Peregrine solution could be obtained by reduction of algebraic solutions for Davey-Stewartson equations (Satsuma & Ablowitz 1979)**

- **These rogue waves are related to homoclinic solutions under a certain limit.**

1. Introduction

Recently, rogue waves in various other integrable systems have also been derived, including

- **Hirota equation (Akhmediev et al. 2010)**
- **derivative NLS equation (He et al. 2011, Guo et al. 2012)**
- **Davey-Stewartson equations (Ohta & Yang, 2012, 2013)**
- **Manakov equations (Degasperis et al. 2012)**
- **Three-wave interaction model (Degasperis et al. 2013)**
- **Maxwell-Bloch equations (He et al. 2013)**
- **…………..**

  Yajima-Oikawa system (Feng, Maruno, 2015)

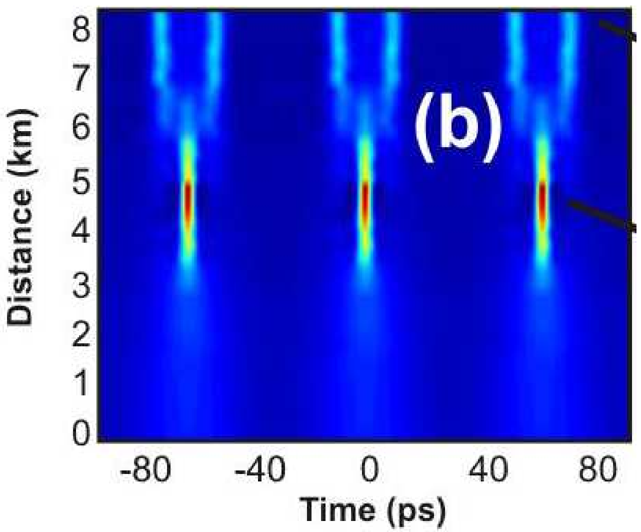
Main techniques used:

- **Darboux transformation**
- **Bilinear methods**

1. Introduction

Experiments:

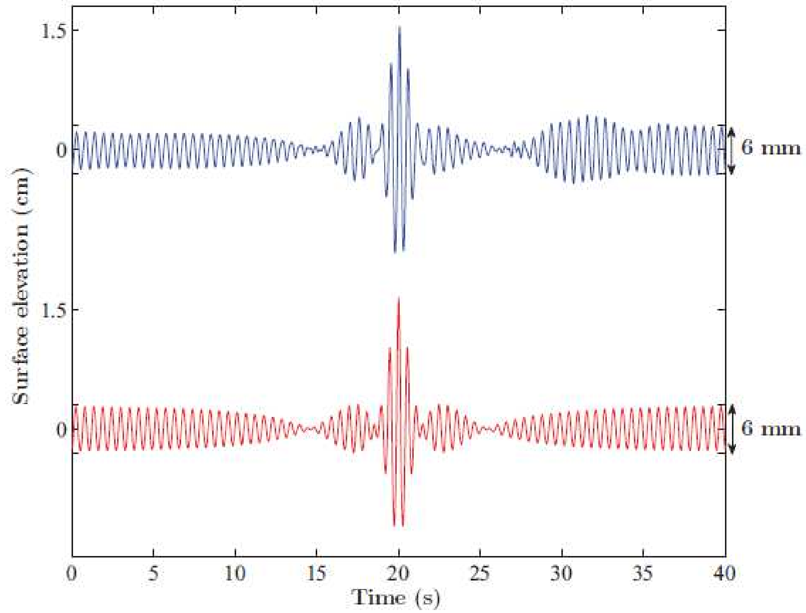
Optical fibers



1st-order rogue wave

Kibler, et al. 2010

Water tanks



2nd-order rogue wave

Chabchoub, et al. 2012

1. Introduction

In this talk, we study rogue waves in the discrete Ablowitz-Ladik equations.

Ablowitz-Ladik equations are the first space-discrete integrable equations discovered by Ablowitz and Ladik in 1976.

Special first-order and second-order rogue waves in the focusing Ablowitz-Ladik equations were reported by Akhmediev and collaborators in 2010.

In this talk, we derive general rogue-wave solutions in both the focusing and defocusing AL equations (Ohta and Yang 2014).

2. Modulation instability in Ablowitz-Ladik equations

Ablowitz-Ladik equations have two types, focusing and defocusing (Ablowitz and Ladik 1976):

Focusing:

$$i\frac{d}{dt}u_n = (1 + |u_n|^2)(u_{n+1} + u_{n-1}), \tag{1}$$

Defocusing:

$$i\frac{d}{dt}u_n = (1 - |u_n|^2)(u_{n+1} + u_{n-1}). \tag{2}$$

First we study modulation instability in the AL equations.

Why? Because modulation instability is precursor of rogue waves.

2. Modulation instability in Ablowitz-Ladik equations

Modulation instability in the focusing AL equation

The constant-background solution is

$$u_n(t) = re^{-2i(1+r^2)t}.$$

To study its modulation instability, we perturb it by normal modes

$$u_n(t) = e^{-2i(1-r^2)t}\left(r + fe^{\lambda t+i\beta n} + \bar{g}e^{\bar{\lambda}t-i\beta n}\right), \qquad (3)$$

where λ and β are the growth rate and wavenumber of the perturbation, and $f, g \ll 1$.

The equation for the growth rate λ is

$$\lambda^2 = 4(r^2+1)(1-\cos\beta)\left[(r^2-1)+(r^2+1)\cos\beta\right].$$

This formula shows that, all constant backgrounds in the focusing AL equation are modulationally unstable.

Thus rogue waves are expected in the focusing AL equation at all background levels.

2. Modulation instability in Ablowitz-Ladik equations

Modulation instability in the defocusing AL equation

Constant-background solution

$$u_n(t) = r\, e^{-2i(1-r^2)t}. \tag{4}$$

Perturbing this solution by normal modes, we obtain the following equation for the growth rate λ:

$$\lambda^2 = 4(r^2-1)(1-\cos\beta)\left[(r^2+1)+(r^2-1)\cos\beta\right]. \tag{5}$$

This formula shows rogue waves with background amplitudes higher than 1 can exist in the defocusing AL equation.

This is surprising! Defocusing nonlinearity supports rogue waves!

A similar phenomenon was reported recently in the defocusing Manakov system by Degasperis et al. (2014).

However, backgrounds lower than 1 are modulationally stable in the defocusing AL, thus no rogue waves can be expected.

3. Formulae for General Rogue Waves in AL Equations

General rogue waves in AL equations are given below:

Theorem 1 General N-th order rogue waves in the Ablowitz-Ladik equations (6)-(7) are given by

$$u_n(t) = \frac{\rho}{\sqrt{1-\rho^2}} \frac{g_n}{f_n} e^{i(\theta n - \omega t)}, \tag{8}$$

where ρ and θ are free real constants, $\omega = 2\cos\theta/(1-\rho^2)$,

$$f_n = \tau_n(0), \quad g_n = \tau_n(1)/(1+\rho)^{2N},$$

$$\tau_n(k) = \det_{1\le i,j\le N}\left(m^{(n)}_{2i-1,2j-1}(k)\right)\bigg|_{p=q=1+\rho},$$

$$m^{(n)}_{ij}(k) = A_i B_j m^{(n)}(k),$$

$$m^{(n)}(k) = \frac{1}{pq-1+\rho^2}(pq)^n \left(\frac{1-\rho^2-q}{1-1/p}\right)^k e^{i\left(\frac{1}{pq}-\frac{1}{1-\rho^2}\right)\left(qe^{i\theta}-pe^{-i\theta}\right)t},$$

3. Formulae for General Rogue Waves in AL Equations

$$A_i = \sum_{\nu=0}^{i} \frac{a_\nu}{(i-\nu)!} [(p-1)\partial_p]^{i-\nu},$$

$$B_j = \sum_{\mu=0}^{j} \frac{\bar{a}_\mu}{(j-\mu)!} [(q-1)\partial_q]^{j-\mu},$$

a_ν are complex constants, overbar $^-$ represents complex conjugation, and

$$a_0 = 1, \quad a_2 = a_4 = \cdots = a_{\text{even}} = 0. \tag{9}$$

When $|\rho| < 1$, these rogue waves satisfy the focusing AL equation (6); and when $|\rho| > 1$, they satisfy the defocusing AL equation (7).

The above expression (8) for rogue waves involves differential operators A_i and B_j. A more explicit and purely algebraic expression for these rogue waves (without the use of such differential operators) is also available, but details are omitted.

3. Formulae for General Rogue Waves in AL Equations

Regarding boundary conditions of these rogue waves at large times, we have the following theorem.

Theorem 2 As $t \to \pm\infty$, solutions $u_n(t)$ in Theorems 1 and 2 approach a constant background,

$$u_n(t) \to (-1)^N \frac{\rho}{\sqrt{1-\rho^2}}\, e^{i(\theta n - \omega t)} \tag{10}$$

uniformly for all n as long as $\cos\theta \neq 0$.

This theorem confirms that solutions $u_n(t)$ in Theorems 1 and 2 are indeed rogue waves, i.e., they rise from a constant background and then retreat back to this same background.

3. Formulae for General Rogue Waves in AL Equations

Regarding regularity (boundedness) of these rogue waves, we have the following theorem.

Theorem 3 General rogue-wave solutions to the focusing AL equation (6) (with $|\rho| < 1$) in Theorems 1 and 2 are non-singular for all times.

Note: this theorem only says that rogue waves for the focusing AL equation are bounded.

It does not say rogue waves for the defocusing equation are bounded.

In fact, we will demonstrate that rogue waves for the defocusing AL equation can blow up to infinity in finite time.

3. Formulae for General Rogue Waves in AL Equations

Remark 1 In these rogue-wave solutions,

- ρ controls the background amplitude, and
- θ is the phase gradient of the solution across the lattice, which is related to the moving velocity of the rogue wave.

This free θ parameter was missed in Akhmediev et al's work.

Remark 2 Non-reducible free parameters in these N-th order rogue waves are $\rho, \theta, \mathrm{Re}(a_1)$ and $a_3, a_5, \ldots a_{2N-1}$, totaling $2N+1$ real parameters.

Remark 3 The $2N+1$ irreducible free parameters in these rogue waves of the AL equations is three more than the corresponding number $2N-2$ in the NLS equation.

The reason is that the NLS equation has three invariances which are lacking in the AL equations: spatial-translation invariance, Galilean-transformation invariance, and scaling invariance. These invariances reduce NLS-rogue free parameters by 3.

4. Dynamics of General Rogue Waves in AL Equations

Now we examine dynamics of rogue waves in AL equations.

Fundamental rogue waves

Fundamental rogue waves are obtained by setting $N = 1$. After shifts of t, n, and utilizing phase and time-shift invariances of the AL equations, these fundamental rogue waves can be rewritten as

$$u_n(t) = \frac{\rho}{\sqrt{1-\rho^2}} e^{i(\theta n - \omega t)} \left[1 + \frac{2i\rho^2\omega t - 1}{\rho^2 \left(n + \omega t \tan\theta - n_0\right)^2 + \rho^4\omega^2 t^2 + \frac{1}{4}(1-\rho^2)} \right],$$

where ρ, θ and n_0 are free real parameters.

The background amplitude is

$$r = \frac{|\rho|}{\sqrt{|1-\rho^2|}}.$$

4. Dynamics of General Rogue Waves in AL Equations

Focusing case

In this case, $|\rho| < 1$, $0 < r < \infty$, and the wave is always bounded.

Non-traveling rogue waves ($\theta = 0$)

$$u_n(t) = \frac{\rho}{\sqrt{1-\rho^2}} e^{-i\omega_0 t} \left[1 + \frac{2i\rho^2\omega_0 t - 1}{\rho^2 (n-n_0)^2 + \rho^4\omega_0^2 t^2 + \frac{1}{4}(1-\rho^2)} \right].$$

where $\omega_0 \equiv 2/(1-\rho^2)$.

(1) on-site rogue waves: $n_0 = 0$

Peak amplitude: $u_{\max} = r(3 + 4r^2)$,

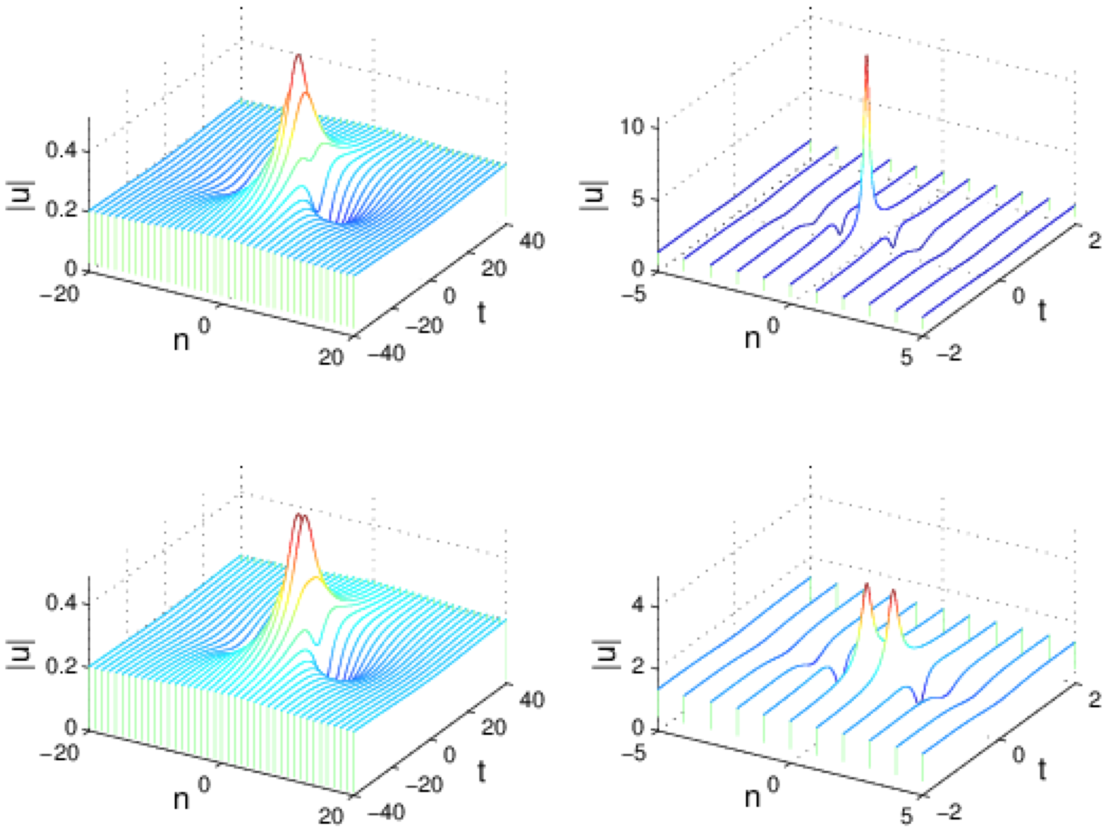
which is at least three times the background amplitude r, and can be much higher when the background is high.

(2) off-site rogue waves: $n_0 = 1/2$

Peak amplitude: $u_{\max} = 3r$, like in the NLS case.

4. Dynamics of General Rogue Waves in AL Equations

Non-traveling fundamental rogue waves in focusing AL



On-site

Off-site

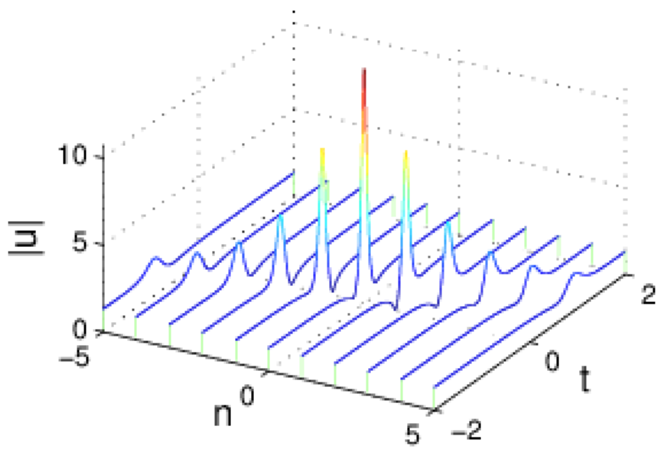
Broad wave **Narrow wave**

4. Dynamics of General Rogue Waves in AL Equations

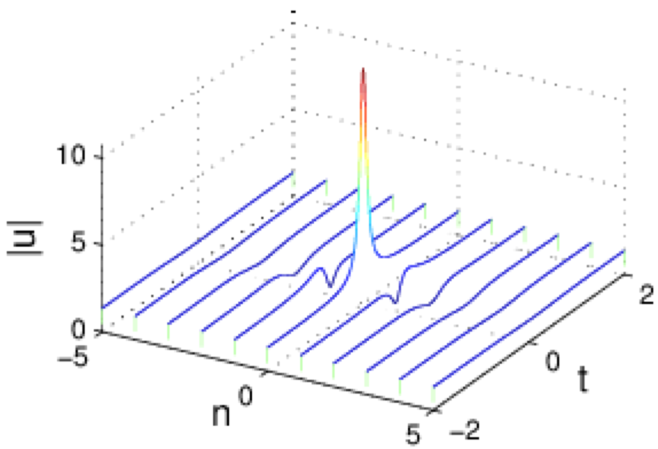
Focusing case

Traveling rogue waves: $\theta \neq 0$

$\theta = -1.2$



$\theta = -0.2$



4. Dynamics of General Rogue Waves in AL Equations

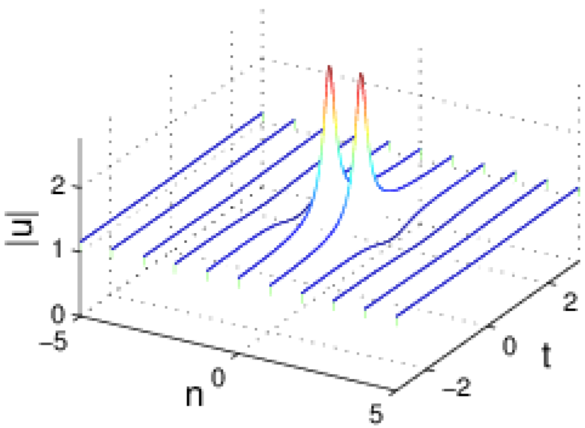
Defocusing case $(|\rho| > 1)$

In this case, rogue waves may explode to infinity in finite time.

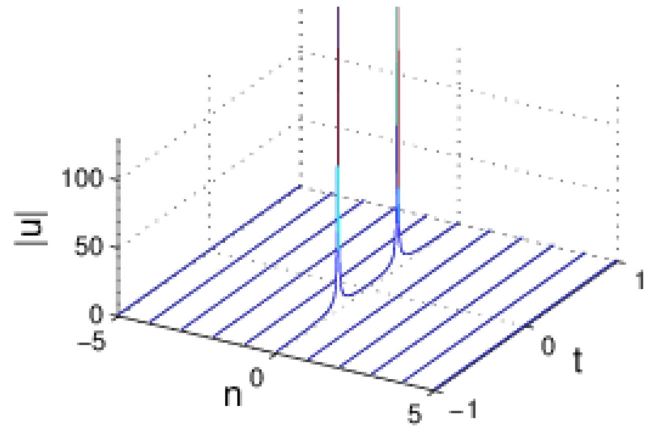
When $\theta = 0$, the blowup condition is

$$|n_0| < \frac{1}{2r},$$

$n_0 = 1/2$

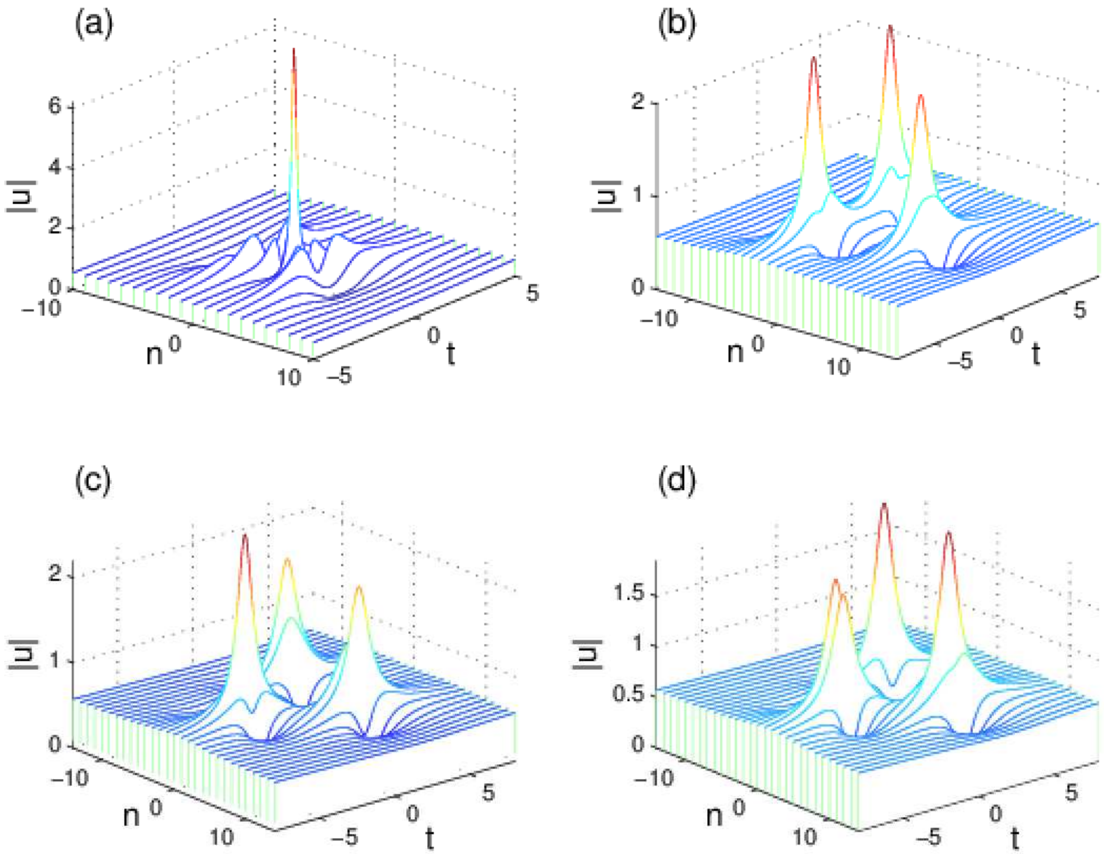


$n_0 = 0$



4. Dynamics of General Rogue Waves in AL Equations

Second-order rogue waves: focusing

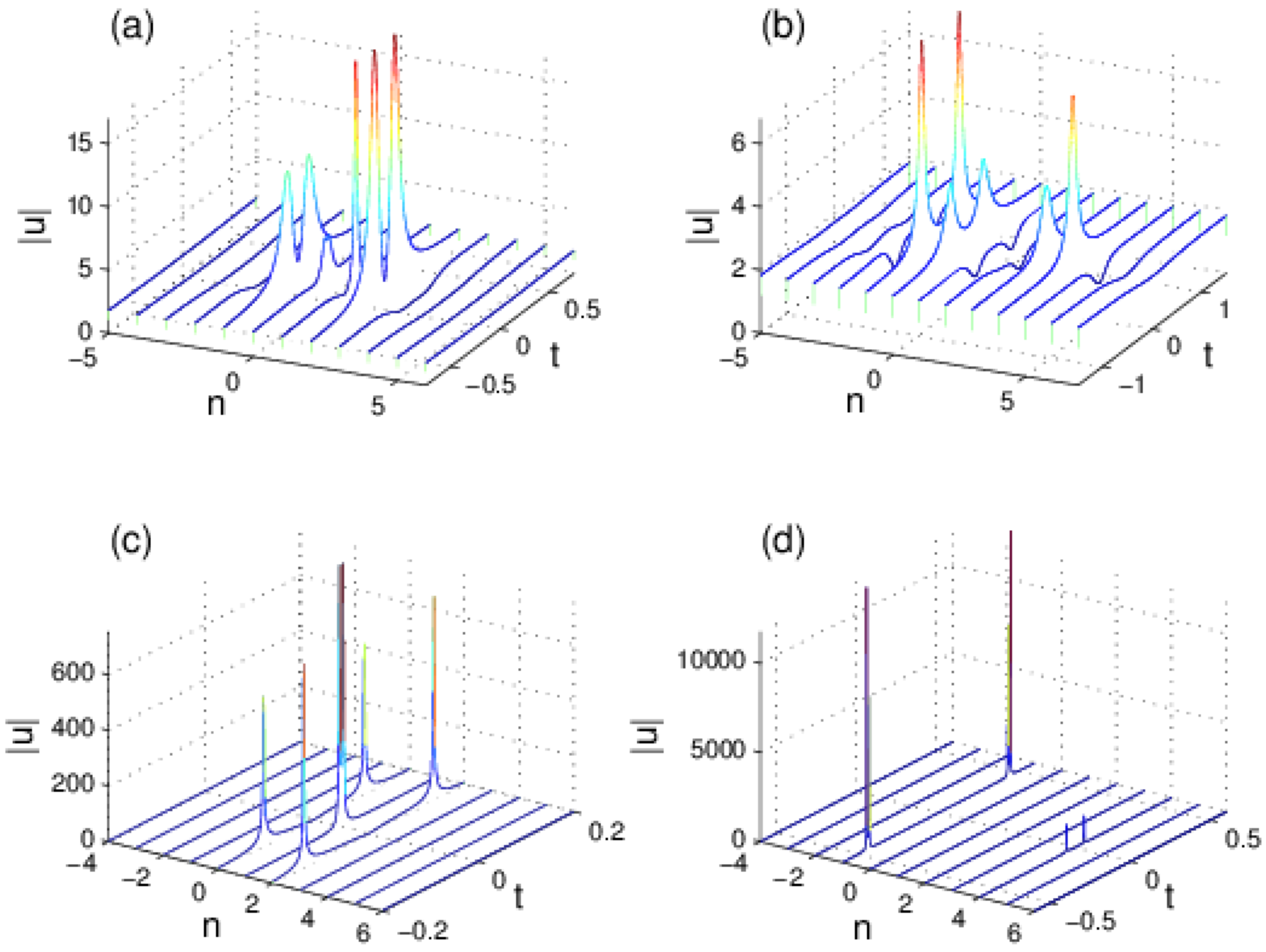


The highest possible peak amplitude is

$$|u|_{\max} = r(5 + 20r^2 + 16r^4).$$

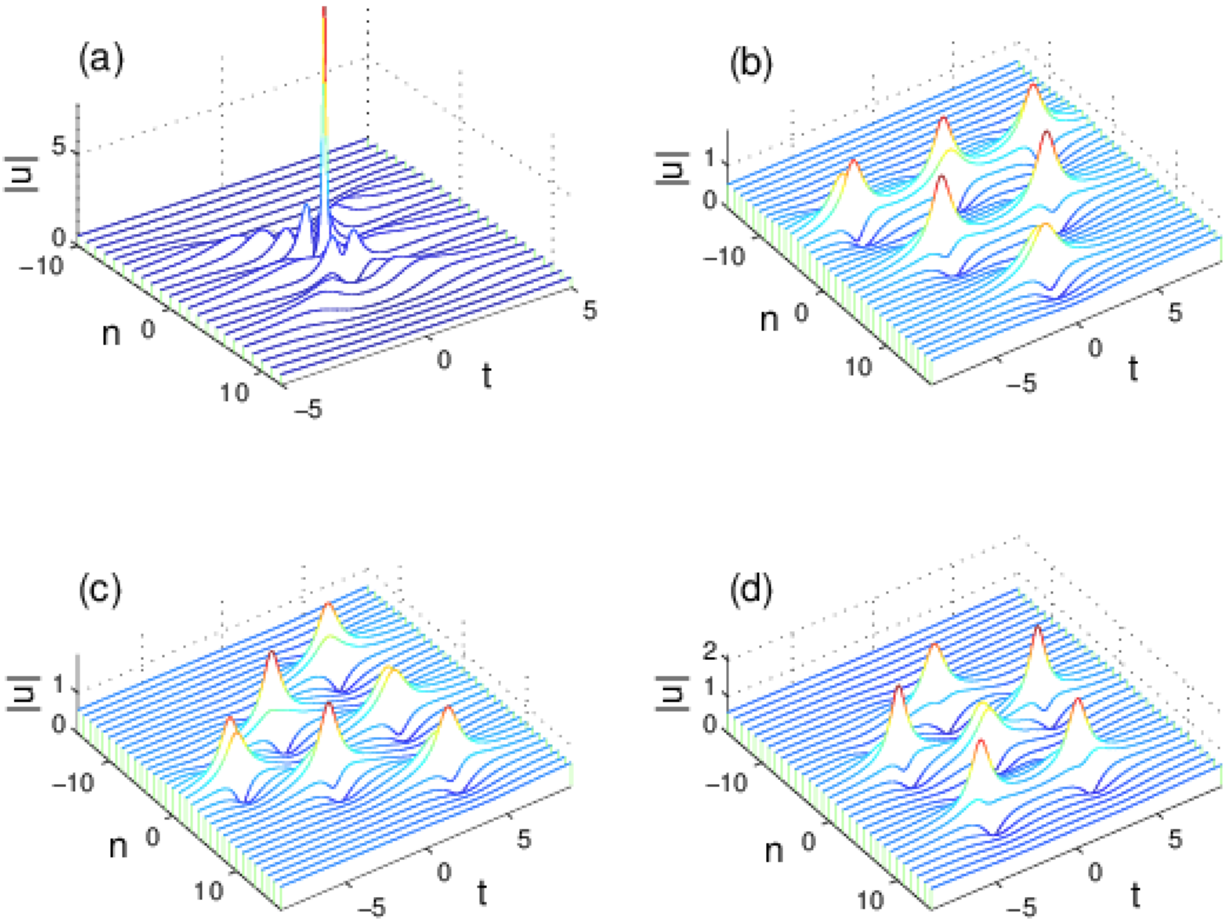
4. Dynamics of General Rogue Waves in AL Equations

Second-order rogue waves: defocusing



4. Dynamics of General Rogue Waves in AL Equations

Third-order rogue waves: focusing



4. Dynamics of General Rogue Waves in AL Equations

Our message:

Rogue waves in focusing AL resemble those in the focusing NLS, but with nontrivial differences such as

- rogue waves in AL can be onsite or offsite, which affect their attainable peak amplitude;
- maximum attainable peak amplitudes of AL rogue waves can be much higher than those in NLS

Rogue waves in defocusing AL have no counterparts in defocusing NLS. These rogue waves may be bounded, but may also blow up to infinity in finite time.

5. Derivation of General Rogue Waves in AL Equations

Now we derive these general rogue-wave solutions in AL equations by the bilinear method.

First, under the variable transformation

$$u_n = \frac{\rho}{\sqrt{1-\rho^2}} \frac{g_n}{f_n} e^{i(\theta n - \omega t)},$$

the AL equations

$$i \frac{d}{dt} u_n = (1 + \sigma |u_n|^2)(u_{n+1} + u_{n-1})$$

become the following bilinear form

$$\begin{aligned} &[i(1-\rho^2) D_t + c + \bar{c}] g_n \cdot f_n = c g_{n-1} f_{n+1} + \bar{c} g_{n+1} f_{n-1}, \\ &f_{n+1} f_{n-1} - (1-\rho^2) f_n f_n = \rho^2 g_n \bar{g}_n, \end{aligned}$$

where $c = e^{-i\theta}$, $\sigma = \mathrm{sgn}(1-\rho^2)$, the overbar represents complex conjugation, and D_t is the Hirota derivative.

5. Derivation of General Rogue Waves in AL Equations

$$[i(1-\rho^2)D_t + c + \bar{c}]g_n \cdot f_n = c g_{n-1} f_{n+1} + \bar{c} g_{n+1} f_{n-1},$$
$$f_{n+1} f_{n-1} - (1-\rho^2) f_n f_n = \rho^2 g_n \bar{g}_n,$$

Our strategy:

(1) Construct $\tau(k,l)$ solutions for general bilinear equations

$$\begin{aligned}
&(D_x + 1)\tau_n(k-1,l) \cdot \tau_n(k,l) = \tau_{n+1}(k-1,l)\tau_{n-1}(k,l), \\
&(D_y - 1)\tau_n(k,l+1) \cdot \tau_n(k,l) = -\tau_{n-1}(k,l+1)\tau_{n+1}(k,l), \\
&\tau_{n+1}(k-1,l)\tau_{n-1}(k,l+1) - (1-\rho^2)\tau_n(k-1,l)\tau_n(k,l+1) \\
&\qquad\qquad\qquad = \rho^2 \tau_n(k-1,l+1)\tau_n(k,l).
\end{aligned}$$

(2) Further require these $\tau(k,l)$ to satisfy the reduction relation

$$\tau_n(k+1,l+1) \propto \tau_n(k,l).$$

Then by properly defining x and y variables, these $\tau(k,l)$ functions would satisfy the bilinear AL equation.

5. Derivation of General Rogue Waves in AL Equations

To achieve the first step, we present the following lemma.

Lemma 1 Let $m_{ij}^{(n)}$, $\varphi_i^{(n)}$ and $\psi_j^{(n)}$ satisfy the following dispersion relations,

$$
\begin{aligned}
&\partial_x m_{ij}^{(n)}(k,l) = \varphi_i^{(n)}(k,l)\psi_j^{(n-1)}(k,l),\\
&\partial_y m_{ij}^{(n)}(k,l) = \varphi_i^{(n-1)}(k,l)\psi_j^{(n)}(k,l),\\
&m_{ij}^{(n+1)}(k,l) = (1-\rho^2)m_{ij}^{(n)}(k,l) + \varphi_i^{(n)}(k,l)\psi_j^{(n)}(k,l),\\
&m_{ij}^{(n)}(k+1,l) = (1-\rho^2)m_{ij}^{(n)}(k,l) - \varphi_i^{(n-1)}(k+1,l)\psi_j^{(n)}(k,l),\\
&m_{ij}^{(n)}(k,l+1) = (1-\rho^2)m_{ij}^{(n)}(k,l) - \varphi_i^{(n)}(k,l)\psi_j^{(n-1)}(k,l+1),\\
&\partial_x \varphi_i^{(n)}(k,l) = \varphi_i^{(n+1)}(k,l),\\
&\partial_y \varphi_i^{(n)}(k,l) = -(1-\rho^2)\varphi_i^{(n-1)}(k,l),\\
&\varphi_i^{(n)}(k-1,l) = \varphi_i^{(n)}(k,l) - \varphi_i^{(n-1)}(k,l),\\
&\varphi_i^{(n)}(k,l+1) = (1-\rho^2)\varphi_i^{(n)}(k,l) - \varphi_i^{(n+1)}(k,l),\\
&(\partial_x+1)\psi_j^{(n)}(k,l) = -\psi_j^{(n-1)}(k+1,l),\\
&(\partial_y-1)\psi_j^{(n)}(k,l) = \psi_j^{(n+1)}(k,l-1),\\
&\psi_j^{(n)}(k+1,l) = (1-\rho^2)\psi_j^{(n)}(k,l) - \psi_j^{(n+1)}(k,l),\\
&\psi_j^{(n)}(k,l-1) = \psi_j^{(n)}(k,l) - \psi_j^{(n-1)}(k,l).
\end{aligned}
\qquad (11)
$$

5. Derivation of General Rogue Waves in AL Equations

Then the determinant

$$\tau_n(k,l) = \det_{1\le i,j\le N}\left(m_{ij}^{(n)}(k,l)\right)$$

satisfies the bilinear equations

$$\begin{aligned}
&(D_x+1)\tau_n(k-1,l)\cdot\tau_n(k,l) = \tau_{n+1}(k-1,l)\tau_{n-1}(k,l),\\
&(D_y-1)\tau_n(k,l+1)\cdot\tau_n(k,l) = -\tau_{n-1}(k,l+1)\tau_{n+1}(k,l),\\
&\tau_{n+1}(k-1,l)\tau_{n-1}(k,l+1) - (1-\rho^2)\tau_n(k-1,l)\tau_n(k,l+1)\\
&\qquad\qquad\qquad\qquad = \rho^2\tau_n(k-1,l+1)\tau_n(k,l).
\end{aligned}$$

This lemma can be proved straightforwardly by using the Jacobi formula of determinants.

Comment: this lemma is very useful for constructing various solutions since the matrix elements can be any functions satisfying the dispersion relations.

For rogue waves, we choose the following polynomial solutions for the matrix elements $m_{ij}^{(n)}(k,l)$.

5. Derivation of General Rogue Waves in AL Equations

Lemma 2 We define matrix elements $m_{ij}^{(n)}$ by

$$m_{ij}^{(n)}(k,l) = A_i B_j m^{(n)}(k,l),$$

$$m^{(n)}(k,l) = \frac{1}{pq-1+\rho^2}(pq)^n \left(\frac{1-\rho^2-q}{1-1/p}\right)^k \left(\frac{1-\rho^2-p}{1-1/q}\right)^l e^{\xi+\eta},$$

$$\xi = px - \frac{1-\rho^2}{p}y, \quad \eta = -\frac{1-\rho^2}{q}x + qy,$$

A_i and B_j are differential operators

$$A_i = \sum_{\nu=0}^{i} \frac{a_\nu}{(i-\nu)!}[(p-1)\partial_p]^{i-\nu}, \qquad B_j = \sum_{\mu=0}^{j} \frac{b_\mu}{(j-\mu)!}[(q-1)\partial_q]^{j-\mu},$$

and a_ν, b_μ are constants. Then for any sequences of indices I_1, $I_2, \cdots, I_N$ and $J_1, J_2, \cdots, J_N$, the determinant,

$$\tau_n(k,l) = \det_{1\le i,j\le N}\left(m_{I_i,J_j}^{(n)}(k,l)\right)$$

satisfies the bilinear equations of Lemma 1.

5. Derivation of General Rogue Waves in AL Equations

Proof It is easy to see that the above $m^{(n)}(k,l)$ and

$$\varphi^{(n)}(k,l) = p^n(1-1/p)^{-k}(1-\rho^2-p)^l e^{\xi},$$

$$\psi^{(n)}(k,l) = q^n(1-\rho^2-q)^k(1-1/q)^{-l} e^{\eta}$$

satisfy the following dispersion relations,

$$\begin{aligned}
&\partial_x m^{(n)}(k,l) = \varphi^{(n)}(k,l)\psi^{(n-1)}(k,l),\\
&\partial_y m^{(n)}(k,l) = \varphi^{(n-1)}(k,l)\psi^{(n)}(k,l),\\
&m^{(n+1)}(k,l) = (1-\rho^2)m^{(n)}(k,l) + \varphi^{(n)}(k,l)\psi^{(n)}(k,l),\\
&m^{(n)}(k+1,l) = (1-\rho^2)m^{(n)}(k,l) - \varphi^{(n-1)}(k+1,l)\psi^{(n)}(k,l),\\
&m^{(n)}(k,l+1) = (1-\rho^2)m^{(n)}(k,l) - \varphi^{(n)}(k,l)\psi^{(n-1)}(k,l+1),\\
&\partial_x \varphi^{(n)}(k,l) = \varphi^{(n+1)}(k,l),\\
&\partial_y \varphi^{(n)}(k,l) = -(1-\rho^2)\varphi^{(n-1)}(k,l),\\
&\varphi^{(n)}(k-1,l) = \varphi^{(n)}(k,l) - \varphi^{(n-1)}(k,l),\\
&\varphi^{(n)}(k,l+1) = (1-\rho^2)\varphi^{(n)}(k,l) - \varphi^{(n+1)}(k,l),\\
&(\partial_x + 1)\psi^{(n)}(k,l) = -\psi^{(n-1)}(k+1,l),\\
&(\partial_y - 1)\psi^{(n)}(k,l) = \psi^{(n+1)}(k,l-1),\\
&\psi^{(n)}(k+1,l) = (1-\rho^2)\psi^{(n)}(k,l) - \psi^{(n+1)}(k,l),\\
&\psi^{(n)}(k,l-1) = \psi^{(n)}(k,l) - \psi^{(n-1)}(k,l).
\end{aligned}$$

5. Derivation of General Rogue Waves in AL Equations

Thus

$$m_{ij}^{(n)}(k,l) = A_i B_j m^{(n)}(k,l),$$

$$\varphi_i^{(n)}(k,l) = A_i \varphi^{(n)}(k,l),$$

$$\psi_j^{(n)}(k,l) = B_j \psi^{(n)}(k,l)$$

satisfy the dispersion relations of Lemma 1.

Consequently the determinant $\tau_n(k,l)$ satisfies the bilinear equations of Lemma 1.

5. Derivation of General Rogue Waves in AL Equations

We now consider the second step: reduction relation.

For this purpose, we have the following lemma.

Lemma 3 The determinant

$$\tau_n(k,l) = \det_{1\le i,j\le N} \left(m_{2i-1,2j-1}^{(n)}(k,l) \right) \Big|_{p=q=1+\rho},$$

where $m_{ij}^{(n)}(k,l)$ is defined in Lemma 2, satisfies the reduction condition

$$\tau_n(k+1,l+1) = (1+\rho)^{4N} \tau_n(k,l).$$

Due to time constraint, this proof is omitted.

5. Derivation of General Rogue Waves in AL Equations

Proof of Theorem 1 (on AL rogue waves)

Since the τ function in Lemma 3 satisfies both the bilinear equations of Lemma 1 and the reduction condition in Lemma 2, it satisfies all the following bilinear equations,

$$
\begin{aligned}
&(D_x+1)\tau_n(k,l)\cdot\tau_n(k+1,l)=\tau_{n+1}(k,l)\tau_{n-1}(k+1,l),\\
&(D_x+1)\tau_n(k,l+1)\cdot\tau_n(k,l)=\tau_{n+1}(k,l+1)\tau_{n-1}(k,l),\\
&(D_y-1)\tau_n(k,l+1)\cdot\tau_n(k,l)=-\tau_{n-1}(k,l+1)\tau_{n+1}(k,l),\\
&(D_y-1)\tau_n(k,l)\cdot\tau_n(k+1,l)=-\tau_{n-1}(k,l)\tau_{n+1}(k+1,l),\\
&\tau_{n+1}(k,l)\tau_{n-1}(k,l)-(1-\rho^2)\tau_n(k,l)\tau_n(k,l)=\frac{\rho^2}{(1+\rho)^{4N}}\tau_n(k+1,l)\tau_n(k,l+1).
\end{aligned}
$$

We now substitute

$$x=\frac{ict}{1-\rho^2},\quad y=-\frac{idt}{1-\rho^2},$$

where c and d are complex constants.

Then the time derivative becomes

$$i(1-\rho^2)\partial_t=-c\partial_x+d\partial_y,$$

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thus we obtain

$$[i(1-\rho^2)D_t + c + d]\tau_n(k+1,l)\cdot\tau_n(k,l)$$
$$= c\tau_{n-1}(k+1,l)\tau_{n+1}(k,l) + d\tau_{n+1}(k+1,l)\tau_{n-1}(k,l),$$
$$[-i(1-\rho^2)D_t + c + d]\tau_n(k,l+1)\cdot\tau_n(k,l)$$
$$= c\tau_{n+1}(k,l+1)\tau_{n-1}(k,l) + d\tau_{n-1}(k,l+1)\tau_{n+1}(k,l),$$
$$\tau_{n+1}(k,l)\tau_{n-1}(k,l) - (1-\rho^2)\tau_n(k,l)\tau_n(k,l) = \frac{\rho^2}{(1+\rho)^{4N}}\tau_n(k+1,l)\tau_n(k,l+1).$$

The determinant solution in Lemma 3 now becomes

$$\tau_n(k,l) = \det_{1\le i,j\le N}\left(A_{2i-1}B_{2j-1}m^{(n)}(k,l)\right)\Big|_{p=q=1+\rho},$$

$$m^{(n)}(k,l) = \frac{1}{pq-1+\rho^2}(pq)^n\left(\frac{1-\rho^2-q}{1-1/p}\right)^k\left(\frac{1-\rho^2-p}{1-1/q}\right)^l e^{i\left(\frac{1}{pq}-\frac{1}{1-\rho^2}\right)(qd-pc)t}.$$

By taking $b_\mu = \bar{a}_\mu$ and $d = \bar{c}$, the conjugacy condition

$$\tau_n(l,k) = \overline{\tau_n(k,l)}$$

is then satisfied.

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Finally we define

$$f_n = \tau_n(0,0), \quad g_n = \tau_n(1,0)/(1+\rho)^{2N},$$

then f_n is real,

$$\tau_n(0,1)/(1+\rho)^{2N} = \bar{g}_n,$$

and the above bilinear equations yield

$$\begin{aligned}
&[i(1-\rho^2)D_t + c + \bar{c}]g_n \cdot f_n = c g_{n-1} f_{n+1} + \bar{c} g_{n+1} f_{n-1}, \\
&f_{n+1} f_{n-1} - (1-\rho^2) f_n f_n = \rho^2 g_n \bar{g}_n.
\end{aligned}$$

Setting $c = e^{-i\theta}$, this is then the bilinear equation of the AL equation.

Thus the above bilinear solutions f_n, g_n give algebraic solutions of the AL equations, which turn out to be rogue waves.

6. Comparison with Rogue waves in DS equations

Rogue waves in AL equations remind us of rogue waves in the Davey-Stewartson (DS) equations.

DS equations were derived by Benney & Roskes (1969) and Davey & Stewartson (1974), and are also called Benney-Roskes-Davey-Stewartson equations.

DS equations are divided into two types, DSI and DSII:

Similarities with AL:

Rogue waves in DSI are always regular (no blowup);
Rogue waves in DSII may be singular (blowup can occur)

Summary

We have derived rogue waves in the AL equations. We have shown that

- Rogue waves in the focusing AL equation can have arbitrary levels of backgrounds, and they are always regular;
- Rogue waves in the defocusing AL equation also exist. These waves exist only when the background is above a certain threshold, and they may explode to infinity in finite time.